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1983 J. Phys. A: Math. Gen. 16 1117

(http://iopscience.iop.org/0305-4470/16/6/006)

J. Phys. A: Math. Gen. **16** (1983) 1117–1124. Printed in Great Britain

Principal orbits and the Yang–Mills–Higgs model

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Received 3 August 1982

Abstract. We give a strong necessary condition for the principal stabiliser of the action of a compact Lie group to have a non-zero centre. An application to the monopole problem in Yang–Mills–Higgs models is discussed.

1. Introduction

In this paper we give a strong necessary condition on any linear representation Δ of a compact Lie group G such that the principal stabiliser of Δ has a non-zero centre. For the unitary and orthogonal groups we then list all representations (reducible and irreducible) which have non-zero centres for their principal stabiliser.

By relating the centre of the Lie algebra of a stabiliser to the second de Rham cohomology of the corresponding orbit we can apply the result mentioned above to the monopole problem in grand unified models. That is, we can then list all such Yang–Mills–Higgs (YMH) models (see e.g. Jaffe and Taubes 1980), with gauge fields valued in the Lie algebra of G and Higgs fields valued in the carrier space of Δ, for which fundamental isolated monopoles are possible in the model (we will elaborate on this intuitive terminology later).

To give the contents of the paper precisely, we first explain the notation we will use (Kobayashi and Nomizu 1969). Let $M=\mathbb{R}^3$ and $P=M\times G$ be a trivial principal fibre bundle over M with G being a simple, compact, connected Lie group. The bundle projection is $\pi: P\ni(x,g)\to x\in M$ and the group action on P is $g\cdot(x,g')=(x,g'g)$, for all $(x,g')\in P$, $g\in G$. Let Δ be a linear, orthogonal representation of G (not necessarily irreducible) on the real finite-dimensional vector space E‡, the scalar product on E being denoted by $(\cdot,\cdot)$, and let $\mathscr{E}=M\times E$ be the vector bundle associated with P, with the group action being defined by Δ. We let $\mathscr{G}$ be the Lie algebra of G and by Ad we mean the adjoint representation of G on $\mathscr{G}$. The representation of $\mathscr{G}$ on E induced by Δ we denote by δ and the adjoint representation of $\mathscr{G}$ we denote by ad. For any $x\in E$, G_x $(<G)$ represents the stabiliser of x, $G(x)$ the orbit through x and $S(x)$ the stratum containing x (Michel 1972, 1979, 1980) $(G(x)\subset S(x)\subset E)$. For the tangent plane to $G(x)$ at x, the tangent plane to $S(x)$ at x and the normal plane to $G(x)$ at x we write: $\mu(x)=x+\mu'(x)$, $\nu(x)=x+\nu'(x)$ and $\kappa(x)=x+\kappa'(x)$, respectively, where $\mu'(x)$, $\nu'(x)$ and $\kappa'(x)$ are vector subspaces of E such that $\mu'(x)\simeq$

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‡ Every finite-dimensional, real linear representation of a compact Lie group is equivalent to an orthogonal representation (Michel 1972, 1979, 1980).

$T_x(G(x))$, $\nu'(x) \simeq T_x(S(x))$ and $\kappa'(x) = \mu'(x)^{\perp}$. The various strata of E may be partially ordered via the partial ordering on the conjugacy classes of subgroups of G. There exists a minimal stratum, S_0, with respect to this ordering and S_0 is open and dense in E. Any orbit in S_0 is an orbit of maximal dimension and is called a principal orbit, and the stabiliser of any point of S_0 is a stabiliser of minimal dimension and called a principal stabiliser. In the YMH model we are given a connection on P (the gauge fields) and a tensorial 0-form ϕ on $\mathscr{E}$ (the Higgs field).

In § 2, we give the relation between the second de Rham cohomology space of an orbit and the centre of the Lie algebra of the corresponding stabiliser. In § 3, we establish a strong necessary condition on δ such that the Lie algebra of the principal stabiliser has a non-zero centre and we then list the principal stabilisers for all representations Δ where G is a unitary or orthogonal group. In § 4, we consider the application of the result of § 3 to the monopole problem in the YMH model.

2. On the second de Rham cohomology of an orbit

Since $G(x)$ may be identified with the coset G/G_x, $H^2_{\mathrm{DR}}(G(x))$ is given if we know $H^2_{\mathrm{DR}}(G/H)$ for any subgroup H of G.

$H^2_{\mathrm{DR}}(G/H)$ may be expressed in terms of algebraic properties of G and H by elementary methods (Spivac 1975). Let $\mathscr{H}$ denote the Lie algebra of H and $C(\mathscr{H})$ its centre ($C(\mathscr{H}) = \{X \in \mathscr{H} | [X, Y] = 0$, for all $Y \in \mathscr{H}\}$), and let H_0 be the subgroup of H which is path connected to the identity and is of the same dimension as H. Then H_0 is a normal subgroup of H and the factor group H/H_0 is a finite group which acts naturally on $C(\mathscr{H})$ through Ad. Writing $C(\mathscr{H})^{H/H_0} = C(\mathscr{H})^H = \{X \in \mathscr{H} | \mathrm{Ad}(h)X = X$, for all $h \in H\}$ we have

$$H^2_{\mathrm{DR}}(G/H) \simeq C(\mathscr{H})^{H/H_0}. \tag{2.1}$$

It follows from equation (2.1) that a necessary condition for $H^2_{\mathrm{DR}}(G/H) \neq 0$ is $C(\mathscr{H}) \neq 0$. It would be desirable to know for all representations Δ what the stabilisers H are such that $C(\mathscr{H}) \neq 0$. This, though, would be too large a project to undertake from the general standpoint and we shall not do so. However, when H is a principal stabiliser, i.e. a stabiliser of a point in the generic stratum, it is possible, quite generally, to decide for which representations $C(\mathscr{H}) \neq 0$, and in the next section we will address this question.

3. On principal orbits

The main result of this section, stated in theorem 3.2, is a strong necessary condition on a representation δ of $\mathscr{G}$ such that the Lie algebra of a principal stabiliser, $\mathscr{H}$, has a non-zero centre $C(\mathscr{H})$.

Since Δ is a real orthogonal representation of G it follows that $\delta(Y)$ is skew-symmetric for all $Y \in \mathscr{G}$, i.e. $(x, \delta(Y)y) = -(\delta(Y)x, y)$ for all $x, y \in E$, and that $\delta(Y)^2$ is a linear self-adjoint transformation for all $Y \in \mathscr{G}$. Noting that $\mathscr{G}$ is a simple Lie algebra and letting tr denote the operation of taking the sum of all the eigenvalues of a self-adjoint linear transformation, we have that

$$d_\delta = \mathrm{tr}\, \delta(X)^2 / \mathrm{tr}\, \mathrm{ad}(X)^2 > 0, \tag{3.1}$$

called the index of the representation δ of $\mathscr{G}$, and defined for any non-zero $X \in \mathscr{G}$, is independent of X (Andreev *et al* 1967)†.

For any subalgebra $\mathscr{H}$ of $\mathscr{G}$ and for all $X \in \mathscr{H}$ we can block diagonalise $\mathrm{ad}(X)$, i.e.

$$\mathrm{ad}(X) = \mathrm{ad}_{\mathscr{H}}(X) \oplus b_{\mathscr{G}}(X) \tag{3.2}$$

corresponding to the decomposition $\mathscr{G} = \mathscr{H} \oplus \mathscr{H}^{\perp}$. Here $\mathrm{ad}_{\mathscr{H}}$ denotes the adjoint representation of $\mathscr{H}$. If $\mathscr{H}$ is the Lie algebra of the stabiliser of a point $x \in E$ then E may be decomposed as

$$E = \mu'(x) \oplus (\nu'(x) \cap \kappa'(x)) \oplus \nu'(x)^{\perp}, \tag{3.3}$$

where $\mu'(x)$, $\nu'(x)$ and $\kappa'(x)$ are the vector spaces defined in § 1. With respect to this decomposition, $\delta(x)$ for each $X \in \mathscr{H}$ may be written in block-diagonal form

$$\delta(X) = \delta_{\mu'(x)}(X) \oplus 0 \oplus \delta_{\nu'(x)^{\perp}}(X) \tag{3.4}$$

where $\delta_{\mu'(x)}(X)y = \delta(X)y$ for all $y \in \mu'(x)$, $\delta_{\nu'(x)}(X)y = \delta(X)y$ for all $y \in \nu'(x)$ and $\delta(X)y = 0$ for all $y \in \nu'(x) \cap \kappa'(x)$.

The linear mapping $\mathscr{H}^{\perp} \ni X \to \delta(X)x \in \mu'(x)$ is a vector space isomorphism ($\mathscr{H}$ being the Lie algebra of G_x). Also, since for any $X \in \mathscr{H}$, $\delta_{\mu'(x)}(X)^2$ and $b_{\mathscr{G}}(X)^2$ are self-adjoint, it follows that $\delta_{\mu'(x)}(X)^2\delta(Y)x = \lambda\delta(Y)x$ iff $\mathrm{ad}(X)^2 Y = \lambda Y$, where $Y \in \mathscr{H}^{\perp}$. Hence we have the following result.

Proposition 3.1. For any $X \in \mathscr{H}$, the Lie algebra of the stabiliser of $x \in E$, $\mathrm{tr}\, \delta_{\mu'(x)}(X)^2 = \mathrm{tr}\, b_{\mathscr{G}}(X)^2$.

Noting that $\nu'(x)^{\perp} = 0$ if x is in the generic stratum of E, we may write

$$d_\delta = \mathrm{tr}\, b_{\mathscr{G}}(Y)^2 / (\mathrm{tr}\, b_{\mathscr{G}}(Y)^2 + \mathrm{tr}\, \mathrm{ad}_{\mathscr{H}}(Y)^2) \tag{3.5}$$

for any non-zero $Y \in \mathscr{H}$ if $\mathscr{H}$ is the Lie algebra of the principal stabiliser G_x (assuming $\mathscr{H} \neq 0$).

Theorem 3.2. If the Lie algebra, $\mathscr{H}$, of a principal stabiliser of the representation δ of $\mathscr{G}$ on E is non-zero, then $d_\delta \leqslant 1$ and $d_\delta < 1$, $= 1$ respectively implies that $\mathscr{H}$ is non-Abelian, or Abelian.

Proof. If $\mathscr{H}$ is non-zero then from equation (3.5) $d_\delta \leqslant 1$. If now $d_\delta < 1$, then from equation (3.5) $\mathrm{tr}\, \mathrm{ad}_{\mathscr{H}}(Y)^2$ is non-zero for any $Y \in \mathscr{H}$, $Y \neq 0$. Thus $[X, Y] \neq 0$ for all $X \in \mathscr{H}$. Finally, if $d_\delta = 1$ then $\mathrm{tr}\, \mathrm{ad}_{\mathscr{H}}(Y)^2 = 0$ for all $Y \in \mathscr{H}$, in which case $[X, Y] = 0$ for all $X, Y \in \mathscr{H}$.

It follows from this theorem that corollary 3.3 is valid.

Corollary 3.3. Let $\mathscr{H}$ be the Lie algebra of a principal stabiliser of the representation δ of $\mathscr{G}$ on E. If $d_\delta = 1$ then $C(\mathscr{H}) = \mathscr{H}$ and if $d_\delta \neq 1$ then $C(\mathscr{H}) = 0$.

Let us make some remarks on the above analysis. If δ is written as a sum of irreducible representations δ_i, i.e. $\delta = \oplus_i \delta_i$ and if d_{δ_i} denotes the index of δ_i then $d_\delta = \Sigma_i d_{\delta_i}$. Thus to know the index of any finite-dimensional representation of a simple

† In fact d_δ is a rational number (Andreev *et al* 1967).

Lie algebra we need know only the index for all its finite-dimensional, irreducible representations. Moreover, if the Lie algebra is semi-simple then we may apply the above analysis to each of its simple components. In the case of the adjoint representation $d_{ad} = 1$, and the Lie algebra of a principal stabiliser (which is equal to its centre) gives a Cartan subalgebra of $\mathscr{G}$. The index d_δ, given by equation (3.1), is defined for a real representation δ. It is possible that δ may be equivalent to a complex representation β, say, in which case a similar expression to equation (3.1) can define a different index d_β, which is related to the former one by $d_\delta = 2d_\beta$ (Andreev *et al* 1967).

Because of the strength of the necessary condition in corollary 3.3, and because of the comments following the corollary, it is reasonable to try, in connection with equation (2.1), to determine all representations for which the Lie algebra of a principal stabiliser, $\mathscr{H}$, has non-zero centre $C(\mathscr{H})$. We do this now for all unitary and orthogonal groups. In addition, we give H, $\mathscr{H}$ and $C(\mathscr{H})^{H/H_0}$.

First we must determine for what real orthogonal, irreducible representations Δ, $d_\delta \leq 1$. Referring to Andreev *et al* (1967) we give this list of representations in table 1. We use Lie algebra terminology. The basic representations ρ_i correspond to a system of primitive roots (Hsiang and Hsiang 1970). (Note that the representations are *real* irreducible.) Since the index of any real representation is equal to the sum of the indices of all its irreducible factors we can, from table 1, determine all real orthogonal representations of the groups involved which have index 1. We list these representations in table 2. From corollary 3.3 we know that only these real orthogonal representations can have $C(\mathscr{H}) \neq 0$, where $\mathscr{H}$ is the Lie algebra of a principal stabiliser, and by equation (2.1) this is necessary for the second de Rham cohomology of a principal orbit to be non-zero. It is straightforward but very tedious (we omit details) to check directly what are the corresponding principal stabilisers for these representations (the paper by Hsiang and Hsiang (1970) is only of partial use for this task)†. In table 2 we also list these principal stabilisers, H, their Lie algebras $\mathscr{H}$ and $C(\mathscr{H})^{H/H_0}$. (T^k denotes the k-dimensional torus and $\mathbb{R}^k$ its Lie algebra.) We have not included in tables 1 and 2 the adjoint representation because, for that case, we always have $d_{ad} = 1$ and the Lie algebra of the principal stabiliser, $\mathscr{H}$, is equal to $C(\mathscr{H})$ and is a Cartan subalgebra.

4. Physical application

We now consider how the results of the previous section may be useful for the monopole problem in YMH models. In order to discuss this we first need to develop the notion of a topological current as the pull back of an element of $H^2_{DR}(G(x))$ under the Higgs field and a deformation retraction.

For any $x \in E$ it is known (Michel 1972, 1979, 1980) that there exists a neighbourhood V_x of x such that $\kappa(x) \cap V_x$ cuts $G(x)$ at x only and that $U_x = G(\kappa(x) \cap V_x)$ $(= \bigcup_{y \in \kappa(x) \cap V_x} G(y))$ is open. The map $r : U_x \to G(x)$, defined by sending every element of $\kappa(x) \cap V_x$ to x and satisfying $r(\Delta(g)y) = \Delta(g)r$, for all $y \in \kappa(x) \cap V_x$ and $g \in G$ is a deformation retraction and U_x is a tubular neighbourhood of $G(x)$ with fibre map r. Moreover, if A is an open set in E such that it contains the orbit of every one of its elements, and if there exists a point x in E and a set U_x of the form specified above with $U_x \supset A$, then A is also a tubular neighbourhood of $G(x)$ with fibre map r. It may

† For the spin groups we have found the article by Hermann (1974) most useful.

Table 1. The indices d_δ, $d_\delta \leqslant 1$, for all irreducible, real orthogonal representations Δ (omitting the adjoint representation) of simple Lie groups of physical interest (for notation, see text).

Type	Representation Δ		Real dimension of E	Group G	Real dimension of G	Index d_δ
A_r: $r \geqslant 1$	$\overset{1}{\circ}-\circ-\circ-\dots-\circ-\circ \oplus \circ-\circ-\dots-\circ-\overset{1}{\circ}$	$\rho_1 \oplus \rho_r$	$2(r+1)$	$SU(r+1)$	r^2+r	$1/(r+1)$
$r=3$	$\circ-\overset{1}{\circ}-\circ$	ρ^2	6	$SO(6)$	15	1/4
$r \geqslant 4$	$\circ-\overset{1}{\circ}-\dots-\circ-\circ \oplus \circ-\circ-\dots-\overset{1}{\circ}-\circ$	$\rho_2 \oplus \rho_{r-1}$	$r(r+1)$	$SU(r+1)$	r^2+2r	$(r-1)/(r+1)$
$r=5$	$2(\circ-\circ-\overset{1}{\circ}-\circ-\circ)$	$2\rho_3$	40	$SU(6)$	35	1
B_r: $r \geqslant 2$	$\overset{1}{\circ}-\circ-\dots-\circ\Leftarrow\circ$	ρ_1	$2r+1$	$SO(2r+1)$	$2r^2+r$	$1/(2r-1)$
$r=2$	$2(\circ\Leftarrow\overset{1}{\circ})$	$2\rho_2$	8	$Spin(5)$	10	1/3
$r=3$	$\circ-\circ\Leftarrow\overset{1}{\circ}$	ρ_3	8	$Spin(7)$	21	1/5
$r=4$	$\circ-\circ-\circ\Leftarrow\overset{1}{\circ}$	ρ_4	16	$Spin(9)$	36	2/7
$r=5$	$\circ-\circ-\circ-\circ\Leftarrow\overset{1}{\circ}$	ρ_5	32	$Spin(11)$	55	4/9
$r=6$	$\circ-\circ-\circ-\circ-\circ\Leftarrow\overset{1}{\circ}$	ρ_6	64	$Spin(13)$	78	8/11
D_r: $r \geqslant 4$	$\overset{1}{\circ}-\overset{1}{\circ}-\dots-\circ<^{\circ}_{\circ}$	ρ_1	$2r$	$SO(2r)$	$2r^2-r$	$1/(2r-2)$
$r=4$	$\circ-\circ<^{\circ^1}_{\circ} \oplus \circ-\circ<^{\circ}_{\circ^1}$	$\rho_3 \oplus \rho_4$	16	$Spin(8)$	28	1/3
$r=5$	$\circ-\circ-\circ<^{\circ^1}_{\circ} \oplus \circ-\circ-\circ<^{\circ}_{\circ^1}$	$\rho_4 \oplus \rho_5$	32	$Spin(10)$	45	1/2
$r=6$	$2(\circ-\circ-\circ-\circ<^{\circ^1}_{\circ})$	$2\rho_5$(or $2\rho_6$)	64	$Spin(12)$	66	4/5

Table 2. We list all real orthogonal representations (reducible or irreducible; omitting the adjoint representation), which have index 1 and that can be constructed from table 1, together with a corresponding stabiliser H (they are all conjugate), its Lie algebra $\mathscr{H}$ and dim $C(\mathscr{H})^{H/H_0}$.

Type	Representation Δ	Group G	$\mathscr{H}$	H	dim $C(\mathscr{H})^{H/H_0}$
A_r: $r \geqslant 1$	$(r+1)(\rho_1 \oplus \rho_r)$	SU$(r+1)$	0		
$r=3$	$4\rho_2$	SO(6)	$\mathbb{R}$	T^1	1
	$3\rho_2 \oplus (\rho_1 \oplus \rho_3)$		0		
	$2\rho_2 \oplus 2(\rho_1 \oplus \rho_3)$		0		
	$\rho_2 \oplus 3(\rho_1 \oplus \rho_3)$		0		
$r \geqslant 4$	$(\rho_2 \oplus \rho_{r-1}) \oplus 2(\rho_1 \oplus \rho_r)$	SU$(r+1)$	0		
$r=5$	$2\rho_3$	SU(6)	$\mathbb{R}^2$	T^2	2
B_r: $r \geqslant 2$	$(2r-1)\rho_1$	SO$(2r+1)$	$\mathbb{R}$	T^1	1
$r=2$	$3(2\rho_2)$	Spin(5)	0		
	$2(2\rho_2) \oplus \rho_1$		0		
	$(2\rho_2) \oplus 2\rho_1$		0		
$r=3$	$5\rho_3$	Spin(7)	0		
	$4\rho_3 \oplus \rho_1$		0		
	$3\rho_3 \oplus 2\rho_1$		0		
	$2\rho_3 \oplus 3\rho_1$		0		
	$\rho_3 \oplus 4\rho_1$		0		
$r=4$	$3\rho_4 \oplus \rho_1$	Spin(9)	0		
	$2\rho_4 \oplus 3\rho_1$		0		
	$\rho_4 \oplus 5\rho_1$		0		
$r=5$	$2\rho_5 \oplus \rho_1$	Spin(11)	0		
	$\rho_5 \oplus 5\rho_1$		0		
$r=6$	$\rho_6 \oplus 3\rho_1$	Spin(13)	0		
D_r: $r \geqslant 4$	$(2r-2)\rho_1$	SO$(2r)$	$\mathbb{R}$	T^1	1
$r=4$	$3(\rho_3 \oplus \rho_4)$	Spin(8)	0		
	$2(\rho_3 \oplus \rho_4) \oplus 2\rho_1$		0		
	$(\rho_3 \oplus \rho_4) \oplus 4\rho_1$		0		
$r=5$	$2(\rho_4 \oplus \rho_5)$	Spin(10)	0		
	$(\rho_4 \oplus \rho_5) \oplus 4\rho_1$		0		
$r=6$	$2\rho_5 \oplus 2\rho_1$	Spin(12)	0		

be remarked that any A above is a tubular neighbourhood of $G(0)=\{0\}$. However, the choice $x=0$ is a trivial one and only cases where $G(x) \neq \{0\}$ will be of interest. In the case where $\Delta = \mathrm{Ad}$ and $E=\mathscr{G}$ the constructution of these tubular neighbourhoods has been given by Houston (1983). In general, the construction of these tubular neighbourhoods will be highly dependent on the representation Δ. However, for an arbitrary representation almost all A are contained in the generic stratum S_0 because it is open and dense in E, and hence any such A is a tubular neighbourhood of any of its orbits (since we may choose $V_x = U_x = S_0$).

Letting $\sigma_0 : M \ni y \to (y, 1) \in P$ define a section of P, we can define the map $\varphi_0 = \varphi \circ \sigma_0 : M \to E$. Let $\tilde{M}$ be a submanifold of M of the same dimension as M such that there exists an open set A in E containing the orbit of all of its elements which is a tubular neighbourhood of $G(x)$, for some $x \in A$, with fibre map r and such that $\varphi_0(\tilde{M}) \subset A$†. By composing the restriction of φ_0 to $\tilde{M}$, $\varphi_0|_{\tilde{M}}$, with r we obtain the map

† For reasonably behaved Higgs fields φ, $\varphi_0^{-1}(S_0)$ is dense in M and such a manifold $\tilde{M} \subset \varphi_0^{-1}(S_0)$ exists.

$\tilde{\varphi} = r \circ \varphi_0|_{\tilde{M}} : \tilde{M} \to G(x)$. The induced mapping on cohomology $\tilde{\varphi}_* : H^2_{\rm DR}(G(x)) \to H^2_{\rm DR}(\tilde{M})$ allows us to pull back the generators of $H^2_{\rm DR}(G(x))$ to define closed two-forms (conserved currents) on $\tilde{M}$. These conserved currents may be considered as generalising the 't Hooft field strength ('t Hooft 1974).

Now let us briefly discuss how the above conserved currents may be used to count the monopoles' charges, i.e. how they may be used to associate intègral charges to a closed two-cycle in $\tilde{M}$ (i.e. a closed, simply connected, oriented, two-dimensional manifold in $\tilde{M}$) giving the topological charges inside it. Let N be a two-cycle in $\tilde{M}$ ($\tilde{M}, A, G(x)$ and $\tilde{\varphi}$ as above); then $[N]$† generates $H_2(N, \mathbb{Z}) \simeq \mathbb{Z}$. The restriction of the map $\tilde{\varphi}$ to $N, \tilde{\varphi}|_N : N \to G(x)$ induces the following mapping on homology $\tilde{\varphi}|_N^* : H_2(N, \mathbb{Z}) \to H_2(G(x), \mathbb{Z})$. By letting $\{[\gamma_i] | i = 1, \ldots, \alpha\}$ be an orthonormal basis for $H^2_{\rm DR}(G(x))$‡ we may associate a set of integers to N using φ by writing

$$n_i(N) = \int_{\tilde{\varphi}|_N^* N} \gamma_i \qquad i = 1, \ldots, \alpha \tag{4.1}$$

where $\alpha = \dim H^2_{\rm DR}(G(x))$. Moreover, the integers $n_i(N)$, $i = 1, \ldots, \alpha$ are unchanged by deforming N, φ to N', φ', as long as $\tilde{\varphi}'|_{N'}[N'] = \tilde{\varphi}|_N[N]$. In the particular case where we deal with finite-energy configurations of the YMH model with the usual sort of Lagrangian being employed, i.e. with a typical spontaneous symmetry-breaking Higgs potential, and where $N = S^2_R = \{x \in M |, |x| = R\}$, for sufficiently large R, the integers $\{n_i(S^2_R)\}$ become part of the boundary condition (Schwarz 1976).

In the case of YMH models with the Higgs field valued in the adjoint representation it is possible to have isolated fundamental point-like monopoles. By this we mean that, if $x_0 \in \mathbb{R}^3$ is an isolated zero of φ, and for all points $x \neq x_0$ in a sufficiently small neighbourhood of x_0, $\varphi(x)$ is valued only in the generic stratum S_0, then we can associate, via equation (4.1), monopole charges to x_0, using a sufficiently small closed two-cycle surrounding x_0. (In fact, the number of such charges will equal the rank of G in this case.) Here the term fundamental refers to the generic stratum and by isolated we are referring to the fact that no lines (or surfaces) of degeneracy of φ (i.e. $\varphi(x) \notin S_0$) leaving x_0 can occur. We can have, then, such fundamental isolated monopoles for the adjoint representation because the second de Rham cohomology of a principal orbit is isomorphic to a Cartan subalgebra. For more general types of YMH models used in grand unified theories the occurrence of fundamental isolated monopoles depends on whether or not the second de Rham cohomology of a principal orbit is zero. With the results of § 3 we can decide which models (i.e. which groups G and representations Δ) can possess fundamental isolated monopoles. In particular for G being a unitary or orthogonal group this information is supplied in tables 1 and 2.

Acknowledgments

I would like to thank V Kac and especially L Michel for much help throughout the course of this work and in particular with § 4. Financial support is gratefully acknowledged from the Institut des Hautes Études Scientifiques (IHES), Bures-sur-Yvette, and CERN, Geneva. Lastly, I have benefitted from discussions with J Bernstein at the IHES and R Coquereaux, M Gunaydin and R Stora at CERN.

† $[\cdot]$ denotes homology or cohomology class as appropriate.

‡ The inner product is defined by integration, i.e. $H_2(\cdot) \times H^2(\cdot)$ $([C], [\gamma] \to \int_C \gamma \in \mathbb{R})$.

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